Lecture 7

Monday, 12 September 2022 12:29 PM

The Computational Complexity of NE

- Importance: economists vs. computer scientists

"If your laptop can't find it, neither can the market" — Kamal Jain

"I'm an economist so I can ignore computational constraints / I'm a computer scientist so I can ignore gravity" — Lance Fortnow

Defn [2NASH]: Given a 2-player finite normal form game, find an equilibrium.

Q. Is 2NASH $\in NP$?

Recall: Given $L \subseteq \{0,1\}^*$, $L \in NP$ if there is a deterministic TM D that for any input (x, y), runs in time $poly(|x|)$ and:

(i) $\forall x \in L, \exists y$: $D(x,y)$ accepts

(ii) $\forall x \notin L, \forall y$, $D(x,y)$ rejects

y "witness / proof" that $x \in L$.

Then how do we write 2NASH as a subset of $\{0,1\}^*$?
If 2NASH = {all 2-player normal form games which have a NE},
then trivially 2NASH $\in NP$, D just has to check if x is a valid 2PNF game.

Actually 2NASH is a search problem, not a decision problem.
We define the class FNP, or Functional NP.
Instead of strings, this class is defined on binary relations.

Defn: A binary relation $P(x,y)$ where $|y| = poly(|x|)$ is in FNP if there is a deterministic poly-time algo that determines if $P(x,y)$ holds

Every language in NP has a corresponding relation in NP: If $L \in NP$, let D be the poly-time algo that determines L.

Then $P(x,y) = 1$ if y is a witness for x, i.e., $D(x,y)$ accepts

$P(x,y) = 0$ o.w.

Eg. SAT $\in NP$

FSAT $= (\phi, \Gamma)$ s.t. $\phi(\Gamma) = T$

FSAT $\in$ FNP

Similarly, 2-NASH $= (\Gamma, \sigma)$ s.t. σ is a NE for Γ

2NASH $\in$ FNP

In fact since we know that every game has a NE, can put this in the smaller class TFNP:

Total Functional NP

Defn: A binary relation $P(x,y) \in$ TFNP if $P(x,y) \in$ FNP and $\forall x \exists y$ s.t. $P(x,y)$ holds

Note that FSAT $\notin$ TFNP

2NASH $\in$ FNP

We can also define the class FP:

Defn: A binary relation $P(x,y) \in$ FP if $\exists$ a deterministic poly-time algo that, given x, finds y s.t. $|y| = poly(|x|)$ & $P(x,y) = 1$ if such y exists, or returns "none".

Theorem: FP = FNP $\Leftrightarrow$ P = NP

(prove yourself)

Now, 2-NASH $\in$ FNP. Can we say that 2-NASH is FNP-complete?

Theorem: If 2NASH is FNP-complete then NP = coNP

coNP: $L \in$ coNP if there is a deterministic TM D that runs in time $poly(|x|)$ and:

$\forall x \notin L, \exists y$: $D(x,y)$ accepts

$\forall x \in L, \forall y$ $D(x,y)$ rejects

UNSAT $\in$ coNP (and is coNP-complete)

Proof of Theorem: 2NASH is FNP-hard $\Rightarrow \exists$ a poly-time reduction from all problems in FNP to 2NASH
$\Rightarrow \exists$ a poly time reduction from FSAT to 2NASH.
$\Rightarrow \exists$ poly-time fns f, g s.t.
(let P be predicate for 2NASH, Q be the predicate for FSAT)

$$\forall x \begin{cases} \exists y: P(f(x), y) = 1 \Rightarrow Q(x, g(y)) = 1 \\ \forall y \; P(f(x), y) = 0 \Rightarrow Q(x, y) = 0 \end{cases}$$

If $\exists$ such f, g, then we claim UNSAT $\in$ NP.
Given a formula ϕ,

(i) if $f(\phi)$ is a valid game, then
$\exists y: P(f(x), y) = 1 \Rightarrow Q(x, g(y)) = 1 \Rightarrow x \in SAT$

(ii) if $f(\phi)$ is not a valid game, then
$\forall y \; P(f(x), y) = 0 \Rightarrow Q(x,y) = 0 \Rightarrow x \notin SAT$

Thus, $f(\phi)$ is a poly-time verifiable certificate! ∎

Ok. So 2-NASH not FNP-complete.
Can it be TFNP-complete?
Unclear if $\exists$ TFNP-complete problems! This is a "semantic" class, while NP is a "syntactic" class.
(kind of vague: $P(x,y)$ is in TFNP if there always exists a certificate y ... how do you show existence?)

Can define subclasses of TFNP, based on proof of existence!

(i) **PPA**: If a graph has an odd degree node, it has another one

(ii) **PLS**: every directed acyclic graph has a sink

(iii) **PPP**: Any fn. mapping n elts. to $n-1$ elts. has at least one collision

(iv) **PPAD**: If a directed graph has a node w/ in-degree $\neq$ out-degree, then it has another one.

Then 2NASH $\in$ PPAD $\subseteq$ PPA, $\subseteq$ PPP

Defn (End of The Line): Directed graph G consists of 2^n nodes. Each has indegree & out-degree ≤ 1. There are 2 Boolean ckts. of size $poly(n)$. P takes as input a node & outputs the predecessor, S takes as input a node & outputs the successor.
Special node 0^n has no predecessor, but has a successor.
Given as input P, S, find another source or a sink in G.
Clearly, EotL $\in$ TFNP.

Defn: Problem Π is in PPAD if it reduces in poly-time to EotL.

Then by Lemke-Howson, 2NASH $\in$ PPAD (but we haven't shown how to pick directions for the edges, so actually we've only shown 2NASH $\in$ PPA).

For now:

Theorem: 2-NASH is PPAD-complete (Chen, Deng, Teng '09)

Theorem: 2D-SPERNER & 2D-BROUWER are PPAD-complete (Chen, Deng '08)

Further,

Theorem: Given a 2-player symmetric normal form game, the following problems are NP-complete:

(i) are there ≥ 2 NE?

(ii) is there a NE where player 1 gets at least λ utility?

(iii) is there a NE where player 1 plays pure strategy s with positive probability?

etc.

[Gilboa & Zemel '89]